

# The basis digraphs of $p$ -schemes

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## Abstract

It is proved that association schemes with bipartite basis graphs are exactly 2-schemes. This result follows from a characterization of  $p$ -schemes for an arbitrary prime  $p$  in terms of basis digraphs.

**Keywords:** association scheme,  $p$ -partite digraph

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# 1 Introduction

Nowadays, the theory of association schemes is usually considered as a generalization of the theory of finite groups. In this sense,  $p$ -schemes introduced in [6] for a prime number  $p$  give a natural analog of  $p$ -groups that enables us, for example, to get the Sylow theorem for association schemes [3]. It is a routine task to extend this notion to coherent configurations (for short, schemes) the special case of which are association schemes (for the exact definitions and notations concerning schemes see section 2). This was done in [4] where algebraic properties of  $p$ -schemes were studied. In this paper we focus on combinatorial features of  $p$ -schemes.

From a combinatorial point of view an association scheme  $\mathcal{C}$  can be thought as a special partition of a complete digraph into spanning subdigraphs satisfying certain regularity conditions. These subdigraphs are called the *basis digraphs* of  $\mathcal{C}$ ; exactly one of them consists in all loops and is called a *reflexive* one. In order to state our main result, we need the following graph-theoretical notion: given an integer  $p > 1$  a digraph  $\Gamma$  is said to be *cyclically  $p$ -partite* if its vertex set can be partitioned into  $p$  nonempty mutually disjoint sets in such a way that if a pair  $(u, v)$  is an arc of  $\Gamma$ , where  $u$  (resp.  $v$ ) belongs to  $i$ -th (resp.  $j$ -th) set, then  $j - i$  is equal to 1 modulo  $p$ , see [1, p.82].

**Theorem 1.1** *Let  $p$  be a prime and  $\mathcal{C}$  an association scheme. Then  $\mathcal{C}$  is a  $p$ -scheme if and only if each non-reflexive basis digraph of  $\mathcal{C}$  is cyclically  $p$ -partite.*

From [4, Theorem 3.4] it follows that a characterization of arbitrary  $p$ -schemes can be reduced to association scheme case in which Theorem 1.1 works. Besides, a digraph  $\Gamma$  is 2-partite if and only if the corresponding undirected loopless graph  $\Gamma'$  is bipartite. (When  $\Gamma$  is a basis digraph of a scheme  $\mathcal{C}$ , we say that  $\Gamma'$  is the *basis graph* of  $\mathcal{C}$ .) Thus we come to the following characterization of 2-schemes.

**Corollary 1.2** *Let  $\mathcal{C}$  be a scheme. Then  $\mathcal{C}$  is a 2-scheme if and only if each basis graph of  $\mathcal{C}$  is bipartite.*

The proofs of Theorem 1.1 and Corollary 1.2 will be given in Section 4. Section 2 contains notations and definitions concerning schemes. In Section 3

we prove several results on  $p$ -schemes which will be used later in the proof of Theorem 1.1.

**Notations.** Throughout the paper  $V$  denotes a finite set.

By a relation on  $V$  we mean any set  $R \subseteq V \times V$ . The smallest set  $X \subseteq V$  such that  $R \subseteq X \times X$  is called the support of  $R$  and is denoted by  $V_R$ . Set  $\Delta(V) = \{(v, v) : v \in V\}$  to be the diagonal relation on  $V$ .

Given  $R, S \subseteq V \times V$  we set  $RS = \{(u, v) \in V \times V : (u, w) \in R \text{ and } (w, v) \in S \text{ for some } w \in V\}$  and call it the product of  $R$  and  $S$ .

Given sets  $X, Y \subseteq V$  and a set  $\mathcal{R}$  of relations on  $V$  we denote by  $\mathcal{R}_{X,Y}$  the set of all nonempty relations  $R_{X,Y} = R \cap (X \times Y)$  with  $R \in \mathcal{R}$ . We write  $\mathcal{R}_X$  and  $R_X$  instead of  $\mathcal{R}_{X,X}$  and  $R_{X,X}$  respectively.

By an equivalence  $E$  on  $V$  we mean an ordinary equivalence relation on a subset of  $V$ . The set of its classes is denoted by  $V/E$ . Given  $X \subseteq V$  we set  $X/E = X/E_X$ . The set of all equivalences on  $V$  is denoted by  $\mathcal{E}_V$ .

Given an equivalence  $E \in \mathcal{E}_V$  and a set  $\mathcal{R}$  of relations on  $V$  we denote by  $\mathcal{R}_{V/E}$  the set of all nonempty relations  $R_{V/E} = \{(X, Y) \in V/E \times V/E : R_{X,Y} \neq \emptyset\}$  where  $R \in \mathcal{R}$ .

By a digraph we mean a pair  $\Gamma = (V, R)$  where  $R$  is a relation on  $V$ . The digraph is called reflexive if  $\Delta(V) \subseteq R$ .

A cycle of length  $n$  is the digraph  $\tilde{C}^n = (V, R)$  where  $V = \{0, \dots, n-1\}$  and  $R$  consists of arcs  $(i, i+1)$ ,  $i \in V$ , with the addition taken modulo  $n$ .

## 2 Schemes

Let  $V$  be a finite set and  $\mathcal{R}$  a partition of  $V \times V$  closed with respect to the permutation of coordinates. Denote by  $\mathcal{R}^*$  the set of all unions of the elements of  $\mathcal{R}$ . A pair  $\mathcal{C} = (V, \mathcal{R})$  is called a *coherent configuration* [2] or a *scheme* on  $V$  if the set  $\mathcal{R}^*$  contains the diagonal relation  $\Delta(V)$ , and given  $R, S, T \in \mathcal{R}$ , the number

$$c_{R,S}^T = |\{v \in V : (u, v) \in R, (v, w) \in S\}|$$

does not depend on the choice of  $(u, w) \in T$ . The elements of  $V$  and  $\mathcal{R}$  are called the *points* and the *basis relations* of  $\mathcal{C}$  respectively. Two schemes  $\mathcal{C}$  and  $\mathcal{C}'$  are called *isomorphic*,  $\mathcal{C} \cong \mathcal{C}'$ , if there exists a bijection between their point sets which preserves the basis relations.

A set  $X \subseteq V$  is called a *fiber* of  $\mathcal{C}$  if the diagonal relation  $\Delta(X)$  is a basis one. Denote by  $\mathcal{F}$  the set of all fibers. Then

$$V = \bigcup_{X \in \mathcal{F}} X, \quad \mathcal{R} = \bigcup_{X, Y \in \mathcal{F}} \mathcal{R}_{X, Y}$$

where the both unions are disjoint. The scheme  $\mathcal{C}$  is called *homogeneous* or *association scheme* if  $|\mathcal{F}| = 1$ . In this case  $\Delta = \Delta(V)$  is a basis relation of it and

$$c_{R, R^T}^\Delta = c_{R^T, R}^\Delta = |R(u)|, \quad R \in \mathcal{R}, \quad u \in V, \quad (1)$$

where  $R^T = \{(u, v) : (v, u) \in R\}$  and  $R(u) = \{v \in V : (u, v) \in R\}$ . In particular, the cardinality of the latter set does not depend on  $u$ . We denote it by  $d(R)$ . Clearly,  $|R| = d(R)|V|$  for all  $R \in \mathcal{R}$ .

By an *equivalence* of the scheme  $\mathcal{C}$  we mean any element of the set  $\mathcal{E} = \mathcal{E}(\mathcal{C}) = \mathcal{R}^* \cap \mathcal{E}_V$ . Given  $E \in \mathcal{E}$  one can construct schemes

$$\mathcal{C}_{V/E} = (V/E, \mathcal{R}_{V/E}), \quad \mathcal{C}_X = (X, \mathcal{R}_X)$$

where  $X \in V/E$ . If the set  $X$  is a fiber of  $\mathcal{C}$ , then obviously  $X \times X \in \mathcal{E}$ , and hence the scheme  $\mathcal{C}_X$  is homogeneous. The equivalence  $E \neq \Delta$  is *minimal* if no other equivalence in  $\mathcal{E} \setminus \{\Delta\}$  is contained in  $E$ , and  $E \neq V \times V$  is called *maximal* if no other equivalence in  $\mathcal{E} \setminus \{V \times V\}$  contains  $E$ . The set of all maximal (resp. minimal) equivalences of  $\mathcal{C}$  is denoted by  $\mathcal{E}_{max}$  (resp.  $\mathcal{E}_{min}$ ). A homogeneous scheme  $\mathcal{C}$  on at least two points is called *primitive* if  $\mathcal{E} = \{\Delta, V \times V\}$ . Clearly,  $E \in \mathcal{E}_{max}$  (resp.  $E \in \mathcal{E}_{min}$ ) if and only if the scheme  $\mathcal{C}_{V/E}$  (resp.  $\mathcal{C}_X$  for some  $X \in V/E$ ) is primitive.

For a homogeneous scheme  $\mathcal{C}$  the set

$$G = \{R \in \mathcal{R} : d(R) = 1\} \quad (2)$$

forms a group with respect to the product of relations. The identity of this group coincides with  $\Delta$ . The order of an element  $R \in G$  equals the sum of all numbers  $d(S)$  where  $S$  is a basis relation of  $\mathcal{C}$  contained in the set

$$\langle R \rangle = \bigcup_{i \geq 0} R^i. \quad (3)$$

A set  $\mathcal{S} \subseteq \mathcal{R}$  is a subgroup of  $G$  if and only if the union of all  $R \in \mathcal{S}$  belongs to  $\mathcal{E}$ . The scheme  $\mathcal{C}$  is called *regular*, if  $G = \mathcal{R}$ .

Given  $R \in \mathcal{R}^*$  the digraph  $\Gamma(\mathcal{C}, R) = (V_R, R)$  is called the *basis digraph* (resp. the *basis graph*) of a scheme  $\mathcal{C}$ , if  $R \in \mathcal{R}$  (resp.  $R = (S \cup S^T) \setminus \Delta$  for some  $S \in \mathcal{R}$ ). In particular, any basis graph of  $\mathcal{C}$  is an undirected loopless graph. From [5, p.55], it follows that the basis digraph of a homogeneous scheme is strongly connected if and only if the corresponding basis graph is connected. This implies that the relation (3) is an equivalence of the scheme  $\mathcal{C}$ . It is easy to see that it is the smallest equivalence on  $V$  containing  $R$ .

### 3 $p$ -schemes

Throughout this section  $p$  denotes a prime number. A scheme  $\mathcal{C} = (V, \mathcal{R})$  is called a  $p$ -scheme if the cardinality of any relation  $R \in \mathcal{R}$  is a power of  $p$  (for more details see [4] a [6]). The class of all  $p$ -schemes is denoted by  $\mathfrak{C}_p$ .

**Theorem 3.1** *Let  $\mathcal{C} \in \mathfrak{C}_p$  be a primitive scheme. Then  $\mathcal{C}$  is a regular and  $|V| = p$ . In particular, any non-reflexive basis digraph of  $\mathcal{C}$  is isomorphic to  $\vec{C}^p$ .*

**Proof.** By the assumption  $\mathcal{C}$  is a homogeneous scheme. Due to (1) this implies that

$$|\Delta| = |V| = \sum_{R \in \mathcal{R}} d(R).$$

Since  $\mathcal{C} \in \mathfrak{C}_p$ , both the left-hand side and each summand in the right-hand side are powers of  $p$ . Taking into account that  $d(\Delta) = 1$  and  $|V| \geq 2$  (because of the primitivity), we conclude that there exists a non-diagonal relation  $R \in \mathcal{R}$  such that  $d(R) = 1$ . By [5, p.71] any primitive scheme having such a basis relation  $R$  is a regular scheme on  $p$  points. ■

A special case of the following statement was proved in [4].

**Theorem 3.2** *Let  $\mathcal{C}$  be a homogeneous scheme,  $E \in \mathcal{E}$  and  $X \in V/E$ . Then  $\mathcal{C} \in \mathfrak{C}_p$  if and only if  $\mathcal{C}_{V/E} \in \mathfrak{C}_p$  and  $\mathcal{C}_X \in \mathfrak{C}_p$ .*

**Proof.** The necessity follows from the obvious equality

$$|R| = |R_{V/E}| \cdot |R_{X,Y}|, \quad R \in \mathcal{R}, \quad (4)$$

where  $X, Y \in V/E$  with  $R_{X,Y} \neq \emptyset$ . Let us prove the sufficiency. Without loss of generality we may assume that  $|V| > 1$ . Suppose that  $E \notin \mathcal{E}_{min}$ . Then there exists an equivalence  $F \in \mathcal{E}_{min}$  such that  $F \subsetneq E$ . The scheme  $\mathcal{C}' = \mathcal{C}_{V/F}$  is a homogeneous one and by [6, Theorem 1.7.6] we have

$$\mathcal{C}'_{V'/E'} \cong \mathcal{C}_{V/E}, \quad \mathcal{C}'_{X'} \cong (\mathcal{C}_X)_{X/F},$$

where  $V' = V/F$ ,  $E' = E_{V/F}$  and  $X'$  is the class of  $E'$  such that  $X/F = X'$ . From the first part of the proof (for  $\mathcal{C} = \mathcal{C}_X$  and  $E = F_X$ ) it follows that  $(\mathcal{C}_X)_{X/F} \in \mathfrak{C}_p$ . Since  $\mathcal{C}_{V/E} \in \mathfrak{C}_p$  and  $|V'| < |V|$ , we conclude by induction that  $\mathcal{C}_{V/F} = \mathcal{C}' \in \mathfrak{C}_p$ . Thus we can replace  $E$  by the equivalence  $F \in \mathcal{E}_{min}$ . In this case the scheme  $\mathcal{C}_X$  is a primitive  $p$ -scheme. By Theorem 3.1 it is a regular scheme on  $p$  points. Thus  $\mathcal{C} \in \mathfrak{C}_p$  by [4, Theorem 3.2]. ■

A set  $X \subseteq V$  is called a *block* of a scheme  $\mathcal{C}$  if there exists an equivalence  $E \in \mathcal{E}$  such that  $X \in V/E$ . Clearly,  $V$  is a block; it is called a *trivial* one. Denote by  $\mathcal{B}$  the set of all nontrivial blocks of  $\mathcal{C}$ .

**Theorem 3.3** *Let  $\mathcal{C}$  be a homogeneous scheme satisfying the following conditions:*

- (1)  $|\mathcal{E}_{max}| \geq 2$ ,
- (2)  $\mathcal{C}_X \in \mathfrak{C}_p$  for all  $X \in \mathcal{B}$ .

*Then  $\mathcal{C} \in \mathfrak{C}_p$ .*

**Proof.** First suppose that the scheme  $\mathcal{C}$  is regular. Then it suffices to verify that the group  $G$  defined by formula (2) is a  $p$ -group. However, from condition (2) it follows that any proper subgroup of  $G$  is a  $p$ -group. This implies that  $G$  is a  $p$ -group unless it is of prime order other than  $p$ . Since the latter contradicts to condition (1), we are done.

Suppose that  $\mathcal{C}$  is not regular. By condition (1) there are distinct equivalences  $E_1, E_2 \in \mathcal{E}_{max}$ . Without loss of generality we can assume that there exists an equivalence  $E \in \mathcal{E}_{min}$  such that

$$E \subseteq E_1 \cap E_2. \tag{5}$$

Indeed, if  $E_1 \cap E_2 \neq \Delta$ , then one can take as  $E$  a minimal equivalence of  $\mathcal{C}$  contained in  $E_1 \cap E_2$ . Otherwise,

$$F_i \cap E_j = \Delta, \quad \{i, j\} = \{1, 2\}. \tag{6}$$

where  $F_i$  is a minimal equivalence of  $\mathcal{C}$  contained in  $E_i$ ,  $i = 1, 2$ . From Theorem 3.1 (applied for  $\mathcal{C} = \mathcal{C}_X$  with  $X \in V/F_i$ ) it follows that  $F_1, F_2 \subseteq G$ . Since  $G$  is closed with respect to products of relations, this implies that it contains the subgroup  $F = \langle F_1, F_2 \rangle$ . Moreover,  $F \neq V \times V$ , for otherwise the scheme  $\mathcal{C}$  is regular which contradicts to the assumption. So there exists an equivalence  $F' \in \mathcal{E}_{max}$  such that  $F \subseteq F'$ . We observe that  $E_1 \neq F'$ , for otherwise

$$F_2 \subseteq \langle F_1, F_2 \rangle = F \subseteq F' = E_1$$

which contradicts to (6). Since  $F_1 \subseteq E_1$  and  $F_1 \subseteq F \subseteq F'$ , this shows that inclusion (5) holds for  $E = F_1$  and  $E_2 = F'$ .

From (5) it follows that  $(E_1)_{V/E}$  and  $(E_2)_{V/E}$  are distinct maximal equivalences of the scheme  $\mathcal{C}' = \mathcal{C}_{V/E}$ . In particular,  $|\mathcal{E}'_{max}| \geq 2$  where  $\mathcal{E}' = \mathcal{E}(\mathcal{C}')$ . Besides, any block of  $\mathcal{C}'$  is of the form  $X' = X/E$  for some block  $X$  of  $\mathcal{C}$ . By condition (2) and Theorem 3.2 this shows that

$$\mathcal{C}'_{X'} = (\mathcal{C}_{V/E})_{X'} \cong (\mathcal{C}_X)_{X/E} \in \mathfrak{C}_p.$$

Thus the scheme  $\mathcal{C}'$  satisfies conditions (1) and (2). Since, obviously,  $|V/E| < |V|$ , it follows by induction that  $\mathcal{C}' \in \mathfrak{C}_p$ . By Theorem 3.2 this implies that  $\mathcal{C} \in \mathfrak{C}_p$  and we are done.  $\blacksquare$

It should be remarked that condition (1) in Theorem 3.3 is essential. Indeed, let  $\mathcal{C}$  be the wreath product of a regular scheme on  $p$  points by a regular scheme on  $q$  points where  $p$  and  $q$  are different primes [5, p.45]. Then the set  $\mathcal{E}_{max} = \mathcal{E}_{min}$  consists of a unique equivalence  $E$  such that  $\mathcal{C}_X$  is a regular scheme on  $p$  points for all  $X \in V/E$ . Thus  $\mathcal{C}$  satisfies condition (2), does not satisfy condition (1) and is not a  $p$ -scheme.

## 4 Proofs of Theorem 1.1 and Corollary 1.2

Throughout this section we fix  $n = |V|$  and an integer  $p > 1$ . By the definition given in the introduction a digraph  $\Gamma = (V, R)$  is cyclically  $p$ -partite if and only if the set  $V$  is a disjoint union of nonempty sets  $V_0, \dots, V_{p-1}$  such that

$$R = \bigcup_{r=0}^{p-1} R_{V_r, V_{r+1}} \quad (7)$$

with addition taken modulo  $p$ . In particular,  $n \geq p$ , and  $n = p$  if and only if  $\Gamma$  is isomorphic to a subdigraph of the directed cycle  $\vec{C}^p$ . The following statement shows that the class of cyclically  $p$ -partite digraphs is closed with respect to taking a *disjoint union* where under the disjoint union of digraphs  $(V_i, R_i)$ ,  $i \in I$ , we mean the digraph  $(V, R)$  with  $V$  and  $R$  being the disjoint unions of  $V_i$ 's and  $R_i$ 's respectively.

**Lemma 4.1** *Let  $\Gamma$  be a disjoint union of strongly connected digraphs  $\Gamma_i$ ,  $i \in I$ . Then  $\Gamma$  is cyclically  $p$ -partite if and only if so is  $\Gamma_i$  for all  $i$ .*

**Proof.** The sufficiency is clear. To prove the necessity suppose that  $\Gamma = (V, R)$  and  $V$  is a disjoint union of nonempty sets  $V_0, \dots, V_{p-1}$  for which equality (7) holds. Let us verify that given  $i \in I$  the digraph  $\Gamma_i = (X, S)$  is cyclically  $p$ -partite. We observe that from (7) it follows that  $S(x) \subseteq R(x) \subseteq V_{r+1}$  for all  $x \in V_r \cap X$  and all  $r$ . On the other hand, since  $\Gamma_i$  is a strongly connected digraph, we also have  $S(x) \neq \emptyset$  for all  $x \in X$ . Thus

$$V_r \cap X \neq \emptyset \Rightarrow V_{r+1} \cap X \neq \emptyset, \quad r = 0, \dots, p-1,$$

whence it follows that  $V_r \cap X \neq \emptyset$  for all  $r$ . Since obviously equality (7) holds for  $R = S$  and  $V_r = V_r \cap X$ , we conclude that  $\Gamma_i$  is cyclically  $p$ -partite. ■

Let  $\Gamma = \Gamma(\mathcal{C}, R)$  be a basis digraph of a homogeneous scheme  $\mathcal{C}$ . Then  $\Gamma$  is strongly connected if and only if the graph  $(V, R \cup R^T)$  is connected (see Section 2), or equivalently  $\langle R \rangle = V \times V$ . This implies that in any case the digraph  $\Gamma$  is disjoint union of digraphs  $\Gamma(C_X, R_X)$  where  $X$  runs over the classes of the equivalence  $\langle R \rangle$ . By Lemma 4.1 this proves the following statement.

**Corollary 4.2** *Let  $\mathcal{C}$  be a homogeneous scheme. Then given a non-diagonal basis relation  $R \in \mathcal{R}$  the digraph  $\Gamma(\mathcal{C}, R)$  is cyclically  $p$ -partite if and only if so is the digraph  $\Gamma(\mathcal{C}_X, R_X)$  for all  $X \in V/\langle R \rangle$ .* ■

**Proof of Theorem 1.1.** Let  $\mathcal{C} = (V, \mathcal{R})$  be a homogeneous scheme. Without loss of generality we may assume that  $n > 1$ .

To prove necessity, suppose that  $\mathcal{C} \in \mathfrak{C}_p$ ,  $\Gamma(\mathcal{C}, R)$  is a non-reflexive basis digraph of  $\mathcal{C}$  and  $E = \langle R \rangle$ . If  $E \neq V \times V$ , then  $|X| < n$ , and  $\mathcal{C}_X \in \mathfrak{C}_p$  for all  $X \in V/E$  (statement (1) of Theorem 3.2). By induction this implies that the digraph  $\Gamma(\mathcal{C}, R_X)$  is cyclically  $p$ -partite for all  $X$ , and we are done



by Corollary 4.2. Let now  $E = V \times V$ . Take  $F \in \mathcal{E}_{max}$ . Then  $\mathcal{C}_{V/F}$  is a primitive  $p$ -scheme (Theorem 3.2) and  $R_{V/F}$  is a non-diagonal basis relation of it. By Theorem 3.1 this implies that

$$\Gamma(\mathcal{C}_{V/F}, R_{V/F}) \cong \vec{C}^p.$$

Therefore, the equivalence  $F$  has  $p$  classes, say  $V_0, \dots, V_{p-1}$ , and equality (7) holds for a suitable numbering of  $V_i$ 's. Thus the graph  $\Gamma(\mathcal{C}, R)$  is cyclically  $p$ -partite.

To prove the sufficiency, suppose that each non-reflexive basis digraph of  $\mathcal{C}$  is cyclically  $p$ -partite. Then by Corollary 4.2 so is each non-reflexive basis digraph of  $\mathcal{C}_X$  for all  $X \in \mathcal{B}$ . By induction this implies that

$$\mathcal{C}_X \in \mathfrak{C}_p, \quad X \in \mathcal{B}. \quad (8)$$

So if  $|\mathcal{E}_{max}| \geq 2$ , then  $\mathcal{C} \in \mathfrak{C}_p$  by Theorem 3.3 and we are done. Otherwise,  $\mathcal{E}_{max} = \{F\}$  for some equivalence  $F \in \mathcal{E}$ . Take a relation  $R \in \mathcal{R}$  such that  $R \cap F = \emptyset$ . Then  $\langle R \rangle \not\subseteq F$  and hence  $\langle R \rangle = V \times V$ . In particular, the digraph  $\Gamma = \Gamma(\mathcal{C}, R)$  is strongly connected. Since it is also cyclically  $p$ -partite, there exists an equivalence  $E \in \mathcal{E}_V$  with  $p$  classes  $V_0, \dots, V_{p-1}$  for which equality (7) holds. The strong connectivity of  $\Gamma$  implies that

$$(u, v) \in E \Leftrightarrow d(u, v) \equiv 0 \pmod{p}$$

where  $d(u, v)$  denotes the distance between  $u$  and  $v$  in the graph  $\Gamma$ . It follows that  $E$  is a union of relations  $R^{ip}$  where  $i$  is a nonnegative integer. Therefore  $E \in \mathcal{E}$ . This enables us to define the scheme  $\mathcal{C}_{V/E}$ . Due to (7) we have

$$\Gamma(\mathcal{C}_{V/E}, R_{V/E}) \cong \vec{C}^p.$$

So  $\mathcal{C}_{V/E}$  is a scheme on  $p$  points having basis relation  $R_{V/E}$  with  $d(R_{V/E}) = 1$ . It follows that  $\mathcal{C}_{V/E} \in \mathfrak{C}_p$ . Together with (8) this shows that  $\mathcal{C}$  satisfies the sufficiency condition of Theorem 3.2. Thus  $\mathcal{C} \in \mathfrak{C}_p$ . ■

**Proof of Corollary 1.2.** Let  $R$  be a basis relation of the scheme  $\mathcal{C}$ . It is easy to see that a non-reflexive digraph  $(V, R)$  is cyclically 2-partite if and only if the graph  $(V, R \cup R^T)$  is bipartite. Besides, by [4, Theorem 3.4] a scheme  $\mathcal{C}$  is a  $p$ -scheme if and only if so is the scheme  $\mathcal{C}_X$  for all  $X \in \mathcal{F}$ . Thus Theorem 1.1 implies that  $\mathcal{C}$  is a 2-scheme if and only if the graph  $\Gamma(\mathcal{C}_X, R \cup R^T)$  is bipartite for all  $X \in \mathcal{F}$  and all  $R \in \mathcal{R}_{X,X} \setminus \{\Delta(X)\}$ . This proves the sufficiency. The necessity follows from the fact that given  $R \in \mathcal{R}_{X,Y}$  with distinct  $X, Y \in \mathcal{F}$ , the graph  $\Gamma(\mathcal{C}, R \cup R^T)$  is bipartite. ■

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